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On the motion established by a constant energy explosion followed by an expanding piston

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The paper examines the influence of a piston motion on the gaseous motion set up by an explosion. This is a problem of relevance not only to explosions in which the remnants of the original explosive are present, but also by the application of the hypersonic similarity law, to the steady flow at high Mach number past a blunt-nosed object with increasing thickness. In the vicinity of the piston perturbation theory breaks down and matching techniques have to be employed. A solution is given in terms of a small parameter which essentially depends on the ratio of the speed of the piston to that of the shock. Some particular cases are discussed in which the piston motion plays a similar role to the counter pressure of the undisturbed gas.

1. Introduction

Some progress has been made in the field of high-speed flow research by the judicious use of the hypersonic similarity law which expresses the equivalence of steady flow at high Mach number with unsteady flow in one less space dimension. It is in the spirit of this law that the present paper is written, for, although the paper discusses an unsteady one-dimensional flow problem, in essence it serves as a useful starting point to that of determining the influence of an afterbody on the flow past an otherwise blunt-nosed body of zero thickness. The inclusion of an afterbody of increasing thickness gives rise to an increasing drag. The case when this drag becomes larger than the initial contribution from the blunt nose has been considered by several authors and their work is discussed by Van Dyke (1966); Stewartson & Thompson (1968) have examined numerically the asymptotic approach to the blast-wave solution for large time, adopting as their model the unsteady flow set up by a piston moving in a semi-infinite tube first with constant velocity and then coming instantaneously to rest. Another feature of the problem, which has been the subject of much discussion (see, for instance, Freeman 1965), is the effect of shock stand-off leading to the establishment of an entropy layer near the body. The present theory, though it could take into account the entropy layer, is not applicable to the flow about the nose itself, but rather in a region just away from it where the finite drag of the blunt nose still dominates over that of the afterbody; this latter drag, which is assumed to be initially zero or much smaller than that due to the blunt nose, will, of course, increase in value and eventually dominate and the theory will be no longer applicable. Mirels (1959, 1962) and Mirels & Thornton (1959) have discussed flows associated with slightly perturbed power law shocks and included in their analyses are some of the features which occur in this paper. The same is also true in rather an indirect way of a paper by Sakurai (1956), whose numerical results we use. The use of Sakurai's computations does enable one to make a comparison of the relative influence of the piston with the counter pressure of the undisturbed gas on the shock path. Mirels ignored this latter effect. For the effect of the piston to begin to dominate results show that the ratio of the piston speed to the sound speed of the undisturbed gas must be in the region of five or more. The extent of the overlap of previous and present work will become apparent later, but essentially we shall be concerned with obtaining a uniformly valid solution throughout the entire region of the flow including that in the vicinity of the body where perturbation theory breaks down. The method of 'inner and outer expansions' is used and leads to the establishment of results from which successive higher approximations may be developed. In addition the method does allow bodies other than power lawed (see Hayes & Probstein 1959, Mirels 1962) to be considered.

The problem described in the paper has its counterpart in certain astrophysical situations such as the ejection of matter from erupting stars in the evolution of a planetary nebula with mass outflow from the central star (see, for instance, Mathews 1966). In these problems a knowledge of the density distribution is important since the emission properties of a gas depend generally on (density)^2. This provides another reason for looking more carefully in the present problem at the flow in the inner region, though one must note the inadequacy of the blast wave solution in describing the starting flow from an explosion of finite dimension.

The method of analysis to be employed is similar to the matching technique described by the author in discussing gravitational effects on expanding H II regions (Goldsworthy 1967), the small parameter in the present problem being the ratio of velocity of the piston to that of the shock. The corresponding parameter in the case of steady flow at high Mach number would be the ratio of the slope of the afterbody to that of the shock. In both these cases the parameter will, in general, vary and be a function of either the time or the distance along the body.

Some particular cases where the effect of the piston motion is similar to that of the counter pressure of the undisturbed gas (this is so for power law piston paths in which $R_p \propto t^{2/(n+3)}$, where $n = 0, 1$ or 2 for plane, cylindrical or spherical motion respectively) are considered. In these we make use of Sakurai's (1956) computations on the effect of counter pressure alone on the propagation of a blast wave. Figures are given showing the shock path, the pressure at the piston and the velocity distribution. From these the reader will see that the counter pressure exerts a greater influence when the Mach number of the piston is less than one, but for Mach numbers of two or more the piston motion quickly establishes itself as the dominating influence near the piston.

The theory does not go all the way in following the transition of the flow from blast dominated to piston dominated, but the indications given by it are at least in the right direction.

2. Equations of motion and the basic flow

Denoting the fluid velocity by u, pressure by p, density by ρ, and measuring distance r from the origin of the explosion and t from the time of the explosion, then for a perfect gas with constant ratio γ of specific heats the equations of motion are

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^n}\frac{\partial}{\partial r}(\rho u r^n) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\frac{p}{\rho^\gamma} = 0. \tag{3}$$

For a constant energy point source explosion the similarity solutions of Sedov (1946*a*, *b*) and Taylor (1950) are applicable. On purely dimensional grounds the radius R at time t of the shock is related to the energy E of the explosion and the initial density ρ_i of the surrounding gas, assumed uniform, by

$$R \approx \left(\frac{E}{\rho_i}\right)^{1/(n+3)} t^{2/(n+3)}. \tag{4}$$

If we take this as a typical dimension of the disturbed gaseous region and assume the shock to be sufficiently strong for the fluid velocity to be of the same order of magnitude as the shock velocity, we can write

$$t = \tau t', \quad r = (E/\rho_i)^{1/(n+3)}\tau^{2/(n+3)}r', \quad u = (E/\rho_i)^{1/(n+3)}\tau^{-(n+1)/(n+3)}u',$$
$$p = \rho_i(E/\rho_i)^{2/(n+3)}\tau^{-2(n+1)/(n+3)}p', \quad \rho = \rho_i\rho', \tag{5}$$

where τ is some measure of the time scale involved. In terms of the new variables the equations of motion are left unaltered; for ease in writing we shall drop the prime.

A similarity solution can now be sought. This is well known and we shall not discuss it in any great detail, except to note its behaviour as $r \to 0$. From the ordinary differential equations or from Sedov's analytic solution one can show that

$$u_0 \sim (U/\gamma)(r/R), \quad \rho_0 \sim k_{1n}(r/R)^{(n+1)/(\gamma-1)}, \quad p_0 \sim [(n+3)/2]^2 k_{2n} U^2, \tag{6}$$

where U and R are the non-dimensionalized shock velocity and radius, respectively, k_{1n}, k_{2n} are known numerical constants given by Sedov (1959) and the subscript 0 is used to label the similarity solution as the basic approximation outside some small inner region in which the piston motion assumes equal importance with the explosive process. For a given piston motion the similarity solution becomes less valid as the piston is approached; only in the rather special case when the piston radius $R_p \approx t^{2/(3+n)\gamma}$ is perfect matching possible. Consequently, in order to discuss the motion in the vicinity of the piston, we choose new inner variables by equating the order of magnitude of the fluid velocity as given in expression (6) to that of the piston, so that, if $U_p(\tau)$ is taken as a measure of the typical piston velocity, the non-dimensional radial distance as introduced in (5) and appropriate to the inner region is given by

$$r = \frac{U_p(\tau)\,\tau}{(E/\rho_i)^{1/(n+3)}\,\tau^{2/(n+3)}}\,\bar{r} = \epsilon^{(\gamma-1)/(2\gamma+n-1)}\,\bar{r}, \tag{7}$$

where $\epsilon = (\rho_i U_p(\tau)^{n+3}\,\tau^{n+1}/E)^{(2\gamma+n-1)/(n+3)(\gamma-1)} \ll 1,$

from which it follows using expressions (6) that the flow variables can be written

$$u = \epsilon^{(\gamma-1)/(2\gamma+n-1)}\,\bar{u}, \quad \rho = \epsilon^{(n+1)/(2\gamma+n-1)}\bar{\rho}, \quad p = \bar{p}. \tag{8}$$

By definition the barred quantities are $O(1)$ in the inner region. The quantity ϵ which is the governing parameter, can be interpreted in several ways, in particular it is some power of the ratio of the piston velocity to that of the shock at time τ. If our interest is centred on small time then ϵ is small if the piston expands at a rate not greater than $\tau^{-(n+1)/(n+3)}$. This is the only case considered in the present paper. With the substitution of expressions (7) and (8) the equations of motion in terms of the inner variables become

$$\epsilon\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{r}}\right)+\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{r}} = 0, \tag{9}$$

$$\frac{\partial\bar{\rho}}{\partial t}+\bar{u}\frac{\partial\bar{\rho}}{\partial\bar{r}}+\frac{\bar{\rho}}{\bar{r}^n}\frac{\partial}{\partial\bar{r}}(\bar{u}\bar{r}^n) = 0, \tag{10}$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial\bar{r}}\right)\left(\frac{\bar{p}}{\bar{\rho}^\gamma}\right) = 0. \tag{11}$$

The basic solution in the inner region is obtained by putting $\epsilon = 0$. Continuing to use the subscript 0 to denote the basic approximation, that in the inner region is obtained by putting $\epsilon = 0$; this gives

$$\bar{p}_0 = [(n+3)/2]^2\,k_{2n}\,U^2, \tag{12}$$

$$-\frac{1}{\bar{r}^n}\frac{\partial}{\partial\bar{r}}(\bar{u}_0\bar{r}^n) = \frac{1}{\bar{\rho}_0}\left(\frac{\partial}{\partial t}+\bar{u}_0\frac{\partial}{\partial\bar{r}}\right)\quad \bar{\rho}_0 = \frac{1}{\gamma\bar{p}_0}\frac{d\bar{p}_0}{dt} = \frac{2}{\gamma}\frac{1}{U}\frac{dU}{dt}, \tag{13}$$

yielding

$$\bar{u}_0 = \frac{F(t)}{\bar{r}^n}-\frac{2}{(n+1)\gamma}\frac{\bar{r}}{U}\frac{dU}{dt}. \tag{14}$$

At the piston $\bar{r} = \bar{R}_p(t)$, $\bar{u}_0 = \bar{U}_p(t)$, so that

$$\bar{u}_0 = \left(\frac{\bar{R}_p}{\bar{r}}\right)^n\left\{\bar{U}_p+\frac{2}{(n+1)\gamma}\frac{\bar{R}_p}{U}\frac{dU}{dt}\right\}-\frac{2}{(n+1)\gamma}\frac{\bar{r}}{U}\frac{dU}{dt}. \tag{15}$$

Equation (11) gives

$$\bar{\rho}_0 = \bar{p}_0^{1/\gamma}\,G(\bar{\xi}_0), \tag{16}$$

where $\bar{\xi}_0$ is the Lagrangian co-ordinate found by integrating (15) along a particle path. It is readily shown that

$$\bar{\xi}_0 = (\bar{r}^{n+1}-\bar{R}_p^{n+1})\,U^{(2/\gamma)}. \tag{17}$$

We now encounter our first real difficulty, for the determination of $G(\bar{\xi}_0)$ involves knowing the entropy distribution at some time within the inner region. We shall adopt the strict mathematical, though not altogether physically correct, approach and trace the particle paths to an earlier time t_1, say, and use there the entropy distribution function as given by Sedov (1959). Since particle paths eventually emerge from the inner region this will be a valid procedure for all except those paths lying very near to the piston, unless we take t_1 to be zero. Unfortunately this leads to the perpetuation at the piston of the same singular behaviour of the temperature as that which occurs at the centre of the explosion in the Sedov solution. It has been pointed out that since the temperature gradient becomes large near the centre that conduction would be important there. This has been considered by Korobeinikov (1957) and its inclusion has the effect of levelling the temperature out. The same action could also be attributed to radiation loss as well as other physical processes. Obviously we could use an acceptable temperature distribution worked out on the basis of some theory or experiment,

$\gamma \backslash n$	0	1	2
1·2	0·40	0·78	1·16
1·3	0·61	1·18	1·72
1·4	0·83	1·60	2·35
1·67	1·47	2·85	4·13

TABLE 1

but for the purpose of illustration we use the known Sedov solution. In the case of steady flow one would have to take into account the entropy layer due to the stand-off of the shock away from the nose.

Equation (17) for a particle path can be written

$$\bar r^{n+1} = \bar R_p^{n+1} + \bar\xi_0 U^{-(2/\gamma)}. \tag{18}$$

According to our basic approximation in which we assume ϵ to be small, the piston must not move at a rate greater than $t^{-(n+1)/(n+3)}$. Hence, since $\gamma > 1$ and $U \propto t^{-(n+1)/(n+3)}$, at sufficiently small time t_1 when the shock velocity is U_1 the particle paths are given approximately by

$$\bar r^{n+1} \sim \bar\xi_0 U_1^{-(2/\gamma)} \tag{19}$$

and emerge from the inner region to pass into the region of overlap of the inner and outer flows where

$$\begin{aligned} \bar\rho_0(\bar r, t_1) &\sim [\bar p_0(t_1)]^{(1/\gamma)}[2/(n+3)]^{(2/\gamma)} k_{1n} k_{2n}^{-(1/\gamma)} (\bar r/R_1 U_1^{(2/\gamma)\cdot(\gamma-1)/(n+1)})^{(n+1)/(\gamma-1)}, \\ &\sim [\bar p_0(t_1)]^{(1/\gamma)}[2/(n+3)]^{(2/\gamma)} k_{1n} k_{2n}^{-1/\gamma} (U_1^2 R_1^{n+1})^{-1/(\gamma-1)} \bar\xi_0^{1/(\gamma-1)}, \end{aligned} \tag{20}$$

R_1 being the shock radius at time t_1.

Hence at time t

$$\bar\rho_0 = \bar p_0^{(1/\gamma)}[2/(n+3)]^{(3/\gamma)} k_{1n} k_{2n}^{-(1/\gamma)} (U_1^2 R_1^{n+1})^{-1/(\gamma-1)} [(\bar r^{n+1} - \bar R_p^{n+1}) U^{(2/\gamma)}]^{1/(\gamma-1)}. \tag{21}$$

We shall use this expression right up to the piston ($\bar\xi_0 = 0$), in which case t_1 must be taken to be zero, since we have assumed that $\bar\xi_0 \gg R_p(t_1)^{n+1} U_1^{(2/\gamma)}$. The quantity $R_1^{n+1} U_1^2$ is then determined from the Sedov solution, there being no need to take into account variations of the shock velocity due to the piston motion. The values of $R_1^{n+1}U_1^2$ displayed in table 1 are those computed by Sakurai (1953).

The inner basic solution is now complete and we can go on to determine the second approximation for the outer solution.

3. The second approximation for the outer solution

In terms of the outer variables expression (15) becomes

$$u = -\frac{2}{(n+1)\gamma}\frac{r}{U}\frac{dU}{dt} + \epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}\left\{\bar{U}_p + \frac{2}{(n+1)\gamma}\frac{\bar{R}_p}{U}\frac{dU}{dt}\right\}\left(\frac{\bar{R}_p}{r}\right)^{n+1}, \quad (22)$$

which suggests we look for an outer solution of the form

$$u = u_0 + \epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}u_1 + \ldots, \quad (23)$$

in which for small r

$$u_1 \sim \left\{\bar{U}_p + \frac{2}{(n+1)\gamma}\frac{\bar{R}_p}{U}\frac{dU}{dt}\right\}\left(\frac{\bar{R}_p}{r}\right)^n$$
$$= \frac{U}{n+1}\frac{R}{\bar{R}_p^{n+1}U^{(2/\gamma)}}\frac{d}{dR}(\bar{R}_p^{n+1}U^{(2/\gamma)})\left(\frac{\bar{R}_p}{R}\right)^{n+1}\left(\frac{R}{r}\right)^n. \quad (24)$$

For a piston path having a power-law variation with time, $\bar{R}_p = \bar{K}t^\alpha \propto R^{\frac{1}{2}(n+3)\alpha}$ we require that

$$u_1 \sim U\left(\frac{\alpha(n+3)}{2} - \frac{1}{\gamma}\right)\left(\frac{\bar{R}_p}{R}\right)^{n+1}\left(\frac{R}{r}\right)^n. \quad (25)$$

For large $\bar{r}$ expression (21) can be expanded

$$\bar{\rho}_0 \sim k_{1n}\left(\frac{U^2R^{n+1}}{U_1^2R_1^{n+1}}\right)^{1/(\gamma-1)}\left(\frac{\bar{r}}{R}\right)^{(n+1)/(\gamma-1)}\left\{1 - \frac{1}{\gamma-1}\left(\frac{\bar{R}_p}{\bar{r}}\right)^{n+1} + \ldots\right\}, \quad (26)$$

which, when transformed to the outer variables, becomes

$$\rho \sim k_{1n}\left(\frac{U^2R^{n+1}}{U_1^2R_1^{n+1}}\right)^{1/(\gamma-1)}\left(\frac{r}{R}\right)^{(n+1)/(\gamma-1)}\left\{1 - \frac{1}{\gamma-1}\epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}\left(\frac{\bar{R}_p}{r}\right)^{n+1} + \ldots\right\}, \quad (27)$$

giving the form of the outer solution for the density as

$$\rho = \rho_0 + \epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}\rho_1 + \ldots, \quad (28)$$

with
$$\rho_1 \sim -\frac{1}{\gamma-1}k_{1n}\left(\frac{r}{R}\right)^{(n+1)/(\gamma-1)}\left(\frac{\bar{R}_p}{r}\right)^{n+1} \quad (29)$$

for small r.

In like fashion we expand the pressure

$$p = p_0 + \epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}p_1 + \ldots. \quad (30)$$

When expressions (23), (28) and (30) are substituted in the equations of motion, the appropriate similarity forms being used for u_0, ρ_0 and p_0, and terms of higher order than the first in $\epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}$ neglected, the resulting linear equations have coefficients involving the similarity variable r/R. Following the method adopted by Sakurai (1956), whose work we will refer to more fully later, we write

$$u = U[f^{(0)} + (c/U)^2 f^{(1)} + \dots], \tag{31}$$

$$p = (U^2/\gamma)\,[g^{(0)} + (c/U)^2 g^{(1)} + \dots], \tag{32}$$

$$\rho = h^{(0)} + (c/U)^2 h^{(1)} + \dots, \tag{33}$$

$$\frac{U_1^2 R_1^{n+1}}{U^2 R^{n+1}} = 1 + \lambda\left(\frac{c}{U}\right)^2 + \dots, \tag{34}$$

where $f^{(0)}$, $g^{(0)}$ and $h^{(0)}$ are functions of the similarity variable $x = r/R$ and are given by the Sedov solution for a constant energy explosion, $f^{(1)}$, $g^{(1)}$ and $h^{(1)}$ are also functions of x and represent the perturbations due to the piston motion, λ is some numerical quantity to be determined, and c is a quantity which will be defined more precisely later; for the moment suffice it to say that c can be a function of time and is proportional to $\epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}$. The reason why $(c/U)^2$ has been used rather than some power of ϵ will become apparent later. The reader is asked to note that in the above expressions U and R are the shock velocity and radius accurate up to the second approximation. The above form of the solution is allowable only if the piston follows a path having a power law variation with time. Taking $\bar{R}_p = Kt^{\alpha}$ and noting the behaviour of u_1 for small r (equation (25)) we require that for small x

$$(c/U)^2 f^{(1)} \sim \left(\frac{\alpha(n+3)}{2} - \frac{1}{\gamma}\right)\left(\frac{\bar{R}_p}{R}\right)^{n+1} \frac{\epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}}{x^n}, \tag{35}$$

so that

$$c^2 \propto t^{(\alpha-4)/(n+3)(n+1)}. \tag{36}$$

The equations satisfied by $f^{(0)}$, $g^{(0)}$ and $h^{(0)}$ are those given by Sakurai; those satisfied by $f^{(1)}$, $g^{(1)}$ and $h^{(1)}$ are slightly different since he took c to be a constant. As noted by Sakurai it is convenient to introduce new dependent variables ϕ, ψ, χ defined by

$$f^{(1)} = (x - f^{(0)})\,\phi, \quad g^{(1)} = g^{(0)}\psi, \quad h^{(1)} = h^{(0)}\chi. \tag{37}$$

The perturbation equations then become

$$-(x-f^{(0)})\,\phi_x + \frac{1}{\gamma D}\psi_x = -[2f_x^{(0)} + \tfrac{1}{2}(n+1)\{\alpha(n+3)-3\} - 1]\,\phi + \left[f_x^{(0)} + \frac{n+1}{2(x-f^{(0)})}\right](\chi - \psi) + \frac{n+1}{2}\left\{\frac{\alpha(n+3)}{2} - 1\right\}\frac{f^{(0)}}{x-f^{(0)}}\lambda, \tag{38}$$

$$(x-f^{(0)})(-\phi_x + \chi_x) = (n+1)(\phi + \{\tfrac{1}{2}[\alpha(n+3)] - 1\}\chi), \tag{39}$$

$$(x-f^{(0)})(-\gamma\phi_x + \psi_x) = (n+1)[(\gamma-1)\,\phi + \{\tfrac{1}{2}[\alpha(n+3)] - 1\}(\psi - \lambda)], \tag{40}$$

where $D = (x-f^{(0)})\,h^{(0)}/g^{(0)}$.

If we multiply (39) by $\gamma - 2/\alpha(n+3)$ and subtract (40), the resulting equation can be integrated and yields

$$\frac{2}{\alpha(n+3)}\phi - \psi + \left\{\gamma - \frac{2}{\alpha(n+3)}\right\}\chi + \lambda = E\exp\left[(n+1)\left\{\frac{\alpha(n+3)}{2} - 1\right\}\int_1^x \frac{dx}{x-f^{(0)}}\right], \tag{41}$$

where E is a constant of integration. Elimination of χ from (39) by the use of (41) yields a system of equations for ϕ and ψ. These are best solved by putting

$$\phi = \phi_1 + \lambda\phi_2, \quad \psi = \psi_1 + \lambda\psi_2, \tag{42}$$

which results in two sets of equations not involving λ. The boundary conditions to be satisfied are the inner matching condition (35) and in the case of a strong shock $\phi_1(1) = \phi_2(1) = \psi_1(1) = \psi_2(1) = 0$. We are particularly interested in the behaviour of the solution as $x \to 0$, which, remembering that $f^{(0)} \sim x/\gamma$ there, we find to be given by

$$\phi_1 \sim A_1 x^{-(n+1)}\{1+O(x^\nu)\} - \frac{1}{\gamma-1}\left\{\frac{\alpha(n+3)}{2}-1\right\}B_1\{1+O(x^\nu)\} + O(x^{\nu+\{(n+1)\gamma\}\{\frac{1}{2}[\alpha(n+3)]-1\}/(\gamma-1)}), \tag{43}$$

$$\psi_1 \sim A_1 O(x^{\nu-(n+1)}) + B_1\{1+O(x^\nu)\} + O(x^{\nu+\{(n+1)\gamma\}\{\frac{1}{2}[\alpha(n+3)]-1\}/(\gamma-1)}), \tag{44}$$

where A_1, B_1 are numerical constants determined by satisfying the shock condition and the differential equations and $\nu = (2\gamma+n-1)/(\gamma-1)$; ϕ_2 and ψ_2 are similar in form except that different numerical constants A_2 and B_2 are used and extra constant terms are added, namely

$$\phi_{20} = -\{\tfrac{1}{2}[\alpha(n+3)]-1\}\chi_2, \tag{45}$$

$$\psi_{20} = 1 + (\gamma-1)\chi_2, \tag{46}$$

where

$$\chi_2 = \frac{1-\dfrac{n+1}{2}\dfrac{\gamma}{\gamma-1}\left\{\dfrac{\alpha(n+3)}{2}-2\right\}}{\left[2+\dfrac{n-1}{2}\gamma+\gamma(n+1)\left\{\dfrac{\alpha(n+3)}{2}-2\right\}\right]\left[\dfrac{\alpha(n+1)}{2}-1\right]+(2-\gamma)\left[1+\dfrac{n+1}{2}\dfrac{\gamma}{\gamma-1}\right]}. \tag{47}$$

The quantity λ is now determined by satisfying the inner boundary condition (35). This requires that

$$\frac{\gamma-1}{\gamma}(A_1+\lambda A_2)\left(\frac{c}{U}\right)^2 = \left(\frac{\alpha(n+3)}{2}-\frac{1}{\gamma}\right)\left(\frac{R_p}{R}\right)^{n+1}, \tag{48}$$

where $R_p = \epsilon^{(\gamma-1)/(2\gamma+n-1)}\bar{R}_p$ is the distance moved by the piston in time t and non-dimensionalized according to scheme (5). A_1 and A_2 are known numerical constants and c^2 can be chosen rather arbitrarily: here we define it by

$$\left(\frac{c}{U}\right)^2 = \frac{1}{\gamma-1}\left(\frac{\alpha(n+3)\gamma}{2}-1\right)\left(\frac{R_p}{R}\right)^{n+1}, \tag{49}$$

so that

$$\lambda = -\frac{A_1}{A_2}+\frac{1}{A_2}. \tag{50}$$

Knowing λ we can now find the correction to the shock path using expression (34). Furthermore, we can determine the pressure change in the vicinity of the piston; using (44) and (46) we see that as $x \to 0$

$$\psi_1 \to B_1 + \lambda(B_2 + 1 + (\gamma-1)\chi_2), \tag{51}$$

giving

$$p = \left(\frac{n+3}{2}\right)^2 k_{2n} U^2 \left[1 + \frac{1}{\gamma-1}\left(\frac{\alpha(n+3)\gamma}{2} - 1\right) \times (B_1 + \lambda\{B_2 + 1 + (\gamma-1)\chi_2\}) \left(\frac{R_p}{R}\right)^{n+1} + \ldots\right]. \tag{52}$$

Using (31), (37), and (43) and retaining only the dominant terms, the velocity as $x \to 0$ is found as

$$u = \frac{Ur}{\gamma R}\left[1 + \left(\frac{\alpha(n+3)\gamma}{2} - 1\right)\left(\frac{R_p}{R}\right)^{n+1} \times \left\{\left(\frac{R}{r}\right)^{n+1} - \frac{1}{\gamma-1}\left(\frac{\alpha(n+3)}{2} - 1\right)(B_1 + \lambda\{B_2 + (\gamma-1)\chi_2\})\right\} + \ldots\right]. \tag{53}$$

4. The second approximation in the inner region

The form of expressions (52) and (53) when written in terms of the inner variables suggests that we try the following expansions,

$$\bar{u} = \bar{u}_0 + \epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}\bar{u}_1 + \ldots, \tag{54}$$

$$\bar{p} = \bar{p}_0 + \epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}\bar{p}_1 + \ldots. \tag{55}$$

Substituting these in (9) and comparing the orders of magnitude of the terms occurring, it is easily seen that, provided $(n+1)(\gamma-1)/(2\gamma+n-1) < 1$, which is true for all γ, $n = 0$ or 1, and $\gamma < 4$ for $n = 2$, the pressure still remains independent of r, so that within the inner region it is given by expression (52). If this expression is substituted in the equation obtained by eliminating the density from the continuity and energy equations, and the condition at the piston satisfied, we find the velocity correct up to the second approximation,

$$\bar{u} = \frac{U\bar{r}}{\gamma R}\left\{\left[1 - \frac{\epsilon^{(n+1)(\gamma-1)/(2\gamma+n-1)}}{\gamma-1}\left(\frac{\alpha(n+3)\gamma}{2} - 1\right)\left(\frac{\alpha(n+3)}{2} - 1\right) \times (B_1 + \lambda\{B_2 + (\gamma-1)\chi_2\})\left(\frac{\bar{R}_p}{\bar{R}}\right)^{n+1} + \ldots\right]\left[1 - \left(\frac{\bar{R}_p}{\bar{r}}\right)^{n+1}\right] + \frac{\alpha(n+3)\gamma}{2}\left(\frac{\bar{R}_p}{\bar{r}}\right)^{n+1}\right\}. \tag{56}$$

As a check the reader will notice that this matches expression (53) when $\bar{r}$ is large.

5. Some particular cases and conclusion

In his paper Sakurai discussed the effect of a constant counter pressure of the unshocked gas on the similarity solutions of Sedov and Taylor. He constructed a solution in the form of a power series in the square of the ratio of the sound velocity of the undisturbed gas to the shock velocity; the quantity c is taken to be the sound velocity of the undisturbed fluid and is a constant. From (36) it is seen that c, as we have defined it, is a constant when $\alpha = 4/(n+3)$. These are the particular cases considered in this section and for which Sakurai's computations can be applied directly. They also serve as illustrative examples of how one should proceed in the general case. It is also of interest to note that in these particular cases the effect of counter pressure can be included; this we will do by taking c to be the undisturbed sound speed non-dimensionalized according to scheme (5). In this case the boundary conditions to be applied at the shock are

$$\left.\begin{aligned}\phi_1(1) &= -2/(\gamma-1), \quad \psi_1(1) = -(\gamma-1)/2\gamma, \quad \chi_1(1) = -2/(\gamma-1),\\ \phi_2(1) &= \psi_2(1) = \chi_2(1) = 0.\end{aligned}\right\} \tag{57}$$

The constant E in (41) is then

$$E = \lambda - \frac{3\gamma-1}{2\gamma}\frac{\gamma+1}{\gamma-1}. \tag{58}$$

The quantity of most interest to us is λ and this we seek to determine. Sakurai found that value of λ for which ϕ is finite at the origin. In our case we have to

n	α	A_1	A_2	B_1	B_2
0	$\frac{4}{3}$	$-23{\cdot}5$	$-11{\cdot}0$	$-8{\cdot}0$	$-4{\cdot}5$
1	1	$-29{\cdot}5$	$-15{\cdot}0$	$-12{\cdot}0$	$-7{\cdot}0$
2	$\frac{4}{5}$	$-32{\cdot}5$	$-17{\cdot}0$	$-14{\cdot}0$	$-8{\cdot}0$

TABLE 2

choose λ such that the matching condition (35) is satisfied. Since c has been chosen to be the sound speed, equation (50) no longer serves to determine λ and must be replaced by

$$\lambda = -\frac{A_1}{A_2} + \frac{2\gamma-1}{\gamma-1}\left(\frac{U}{c}\right)^2\left(\frac{R_p}{R}\right)^{n+1}\frac{1}{A_2}. \tag{59}$$

Since $R_p \propto R^2$ and in the first approximation U^2R^{n+1} is constant, λ is truly constant. When the piston velocity is small compared to the sound speed c we recover Sakurai's values for λ.

The constants A_1 and A_2, and also B_1 and B_2, can be determined from Sakurai's computations, though not very accurately since only graphical results for ϕ_1, ϕ_2, ψ_1 and ψ_2 are given in his paper. However, their accuracy is sufficient to allow the method to be indicated. In table 2 the values of the constants thus computed are given for $\gamma = 1{\cdot}4$ and $n = 0$, 1 and 2, the corresponding values of α being $\frac{4}{3}$, 1 and $\frac{4}{5}$, respectively.

For a cylindrical explosion $(U/c)^2(R_p/R)^2 = (U_p/2c)^2$ and for this case the computed shock paths for various values of U_p/c are shown in figure 1 using expansion (34) and expression (59) for λ. It will be seen that the piston motion has very little effect on shock path unless U_p/c is in the region of about 5 or more, the counter pressure of the undisturbed gas accounting for most of the departure from the basic similarity solution illustrated in the figure by the broken curve. A comparison with the numerical work of Mirels can be made here by taking U_p/c infinite. Using the values given in table 2 the shock path in this case works out to be

$$\frac{R}{R_1} = 1 + 0{\cdot}038\left(\frac{R_p}{R_1}\right)^2,$$

the corresponding value of the constant multiplying $(R_p/R_1)^2$ obtained by Mirels is 0·039. At greater distances than those shown in the figure the effect of the piston is enhanced by the diminishing shock velocity, but here the theory is no longer applicable since the shock is becoming weak.

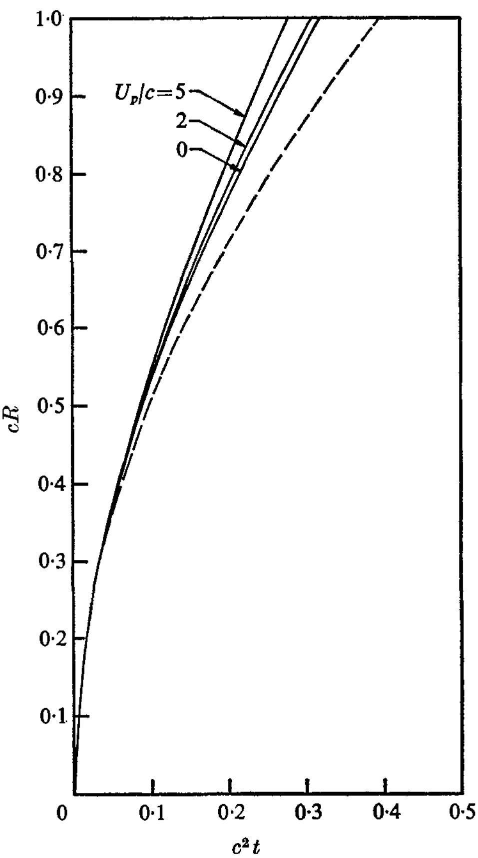

FIGURE 1. Shock paths for $U_p/c = 0$, 2 and 5.

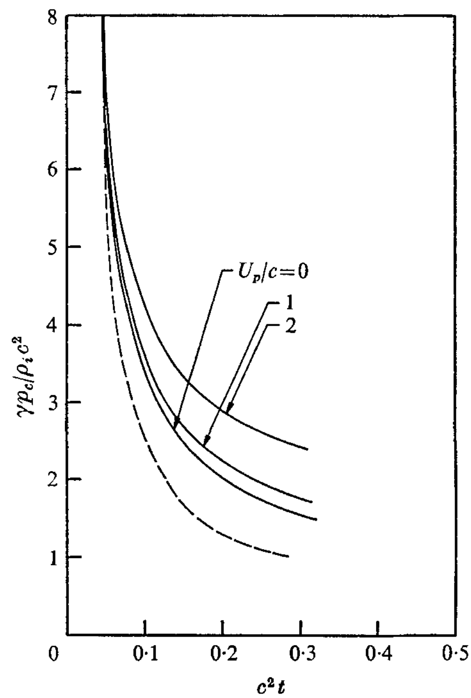

FIGURE 2. The pressure at the piston as a function of c^2t.

The piston motion will have a greater influence in its near vicinity, as can be seen by examining the pressure and the gas velocity. Taking into account the new definition of c, the pressure in the inner region is given by

$$p = \left(\frac{n+3}{2}\right)^2 k_{2n} U^2 \left[1 + \left(\frac{c}{U}\right)^2 (B_1 + \lambda\{B_2 + 1 + (\gamma-1)\chi_2\}) + \dots\right], \tag{60}$$

and the velocity is

$$u = \frac{Ur}{\gamma R}\left\{\left[1 - \left(\frac{c}{U}\right)^2 (B_1 + \lambda\{B_2 + (\gamma-1)\chi_2\}) + \dots\right]\left[1 - \left(\frac{R_p}{r}\right)^{n+1}\right] + 2\gamma\left(\frac{R_p}{r}\right)^{n+1}\right\}. \tag{61}$$

Figure 2 shows the pressure at the piston for $U_p/c = 0$, 1 and 2. Again the pressure of the undisturbed gas exerts most influence for $U_p/c \leqslant 1$, and it is not until the

Mach number of the piston reaches a value of about 2 (for steady flow at Mach number 8, say, this corresponds to cone semi-angle of about 14°) that the piston motion has any noticeable effect.

In the outer region the pressure and velocity are given by expressions (31) and (32) in which Sakurai's tabulated results for $f^{(0)}$, $g^{(0)}$ and his graphical results for ϕ_1, ϕ_2, ψ_1, and ψ_2 can be used. In figure 3 the velocity distribution is shown giving the first and second approximations. The distribution will of course change

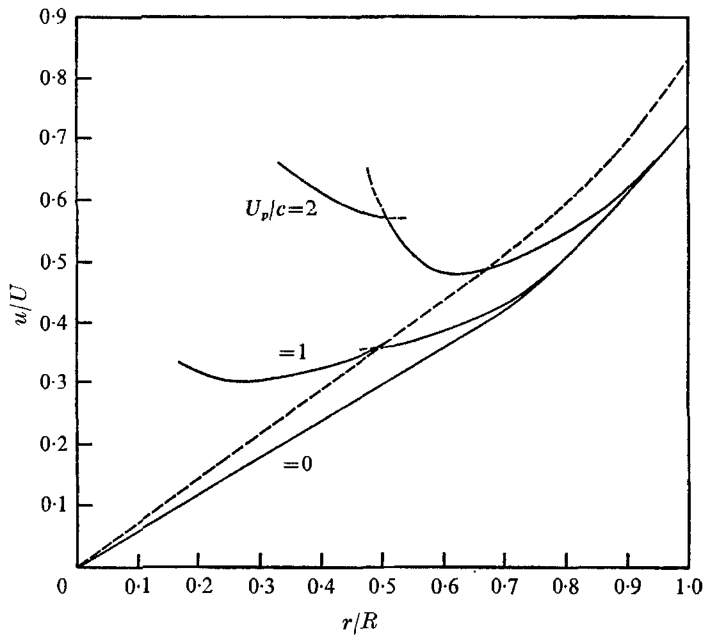

FIGURE 3. The velocity distribution for $U_p/c = 0$, 1 and 2 when $U/c = 3$.

with time and the plot is given at such times when U/c is 3, i.e. when c^2t is about 0·09. The velocity distribution clearly demonstrates the effect of the piston in the inner region and the breakdown of the perturbation solution there. Near the shock its influence is negligible, the counter pressure assuming the more dominant role.

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